A General Theory of the Hydraulic Transport of Solids in Full Suspension

A C Bonapace

A GENERAL THEORY OF THE HYDRAULIC TRANSPORT OF SOLIDS IN FULL SUSPENSION

By A C Bonapace[†]

SYNOPSIS

A theory has been developed following an experimental approach. It covers solids and fluids of any density. It has been verified in detail upon experiments of solid transport in water extracted from the technical literature.

Keywords: Hydraulic transport general theory

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FOREWORD

Dr Antonio C. Bonapace has written a piece which transcends many borders. After finishing his Secondary Education at the Liceo Scientifico in Merano, he completed his Technical Education at the Politecnico of Milan and, in the process, obtained his Doctorate Degree in Mechanical Engineering in June 1948.

As a professional engineer, he held various appointments in the fields of automation and industrial control. Since 1959, his fields of activity included hydraulic transport of solids, combustion and gasification of coals. He was also involved in the study on the abatement of coal dust explosions in mines. Since, he has accomplished a study on the hydraulic transport of solids. His present activity covers the development of hydraulic components for the automation industry.

He has many publications to his credit. But one of his seminal works is this volume entitled “A General Theory of the Hydraulic Transport of Solids in Full Suspension”, and I am proud to introduce it. Here, for your perusal and study, is his work. I leave it all to you, to judge and use it as you wish.

His hobbies include hiking, jogging and mountain climbing. He is one of the oldest Italians to have reached the summit of Mt Blanc of the Alps.

As a parting word, Dr Bonapace hopes that students, researchers and other parties will find his work of interest and all may download his work for free, with his best wishes, provided they acknowledge the source of their ideas.

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NOTATION:

Dimensional		Non-dimensional	
D	$\rightarrow$	Δ°	Pipe diameter
d	$\rightarrow$	δ	particle diameter
f		-	friction factor
g		-	acceleration due to gravity
I		-	hydraulic gradient
k	$\rightarrow$	κ	size of a wall excrescence
L	$\rightarrow$	Λ	a length
RE			stream Reynolds number
Re_*		-	Reynolds number of a particle at the boundary
Sh		-	Shields number referred to the particle diameter
SH		-	Shields number referred to the pipe diameter
V	$\rightarrow$	Ω	mean velocity of the stream
v_*	$\rightarrow$	ω_*	friction velocity of the stream at the boundary
x		-	volumetric solid concentration
α		-	a parameter
β = 1, 11		-	a factor
ε		-	a parameter
η		-	a small quantity (versus 1)
μ			an excess density

ν	-	kinematic viscosity
ρ	-	density of the fluid phase
ρ_s	-	density of the solid phase
suffix		10 means $\frac{\rho_s}{\rho} \to 1$ and $x \to 0$
		lx means $\frac{\rho_s}{\rho} \to 1$ and $x > 0$
		μ 0 means $\mu \neq 0$ and $x \to 0$
		μ x means $\mu \neq 0$ and $x > 0$
		βx means 1,11x

I. INTRODUCTION

In the present paper particle transport by a stream is considered with particles conveyed in a fully suspended state. In a well-known work, Durand and Condolios [2] correlated many experimental results of transport in water, with particles having some intermittent contact with the boundary. The authors however did not define sufficiently the condition by which the considered slurries would proceed with the solid phase fully suspended inside the fluid. It is the object of the present paper to define this “suspension” phenomenon both for a single particle present in the system as well as for many particles transported, i.e. for a certain finite volumetric concentration of the solid material.

The results obtained from the present analysis of sixteen experimental cases of hydraulic transport of particles settling in water have been found to agree with the results of the analysis herein established.

The results could be further extended to particles buoyant in a fluid and to particles transported in gas streams, (e.g. air) i.e. with relative density of the solid phase a few thousand times greater than the fluid.

Hence this presents possibility to forecast the conditions of transport in full suspension for different solids in a great variety of vector fluids.

Operating with non-dimensional quantities further widened the generality of the results. By this the obtained literary expressions resulted independent of the kinematic viscosity of the fluid and of the acceleration due to gravity.

For the description of the various phenomena one has made use of well-known laws of experimental hydraulics. Hence they will only briefly mentioned in this work with reference to the pertinent author.

In the present paper the author could define by elementary methods certain minimum conditions for the energy dissipation function of the stream. Omitting the rather long analytical derivation, these conditions will be given in a very simple analytical form by means of two “experimentally acceptable” relations. One has also defined certain reference conditions associated with an “indifferent” particle, i.e. neither settling nor buoyant in the fluid.

In all the phenomena investigated, be it for a buoyant or settling particle, for great values of the density ratio (as in the conveyance of a solid in a gas) and for great values of the volumetric concentration (as in the conveyance of cylindrical bodies near the size of the pipe), the developed theory could be verified without finding any conceptual inconsistency in these extreme cases of hydraulic transport at the limit.

II. HYDRAULIC PREMISES

One defines below certain well-known expressions related to the flow of streams, using at first dimensional symbols and thereafter non-dimensional ones.

For flow in pipes one denotes a pipe diameter by D, a stream velocity by V, a friction factor by f, the kinematic viscosity by ν, the size of an excrescence on a wall by k, the diameter of a particle by d and the acceleration due to gravity by g.

Denoting by v_* the friction velocity of the stream at the boundary, this is expressed by the well-known relation:

$$v_* = \left(\frac{f}{8}\right)^{\frac{1}{2}} V \tag{2.1}$$

Non-dimensional symbols will now be introduced for physical parameters having dimensions of a velocity and of a length respectively. Considering e.g. a stream velocity V one transforms it into a non dimensional velocity Ω by means of division by the group $(g\nu)^{\frac{1}{3}}$ (dimensionally equivalent to $\frac{m}{s}$) i.e.

$$\frac{V}{(g\nu)^{\frac{1}{3}}} = \Omega \tag{2.2}$$

Analogously for the pipe of diameter D one transforms it to a non-dimensional diameter $\Delta°$ by multiplication by the group $\left(\frac{g}{\nu^2}\right)^{\frac{1}{3}}$ (dimensionally equivalent to $\frac{1}{m}$) i.e.

$$D\left(\frac{g}{\nu^2}\right)^{\frac{1}{3}} = \Delta^\circ \qquad (2.3)$$

In a short notation defining the above transformations by an arrow one represents the transformed non-dimensional quantity by means of a letter of the Greek alphabet. Hence one will write:

For velocities:

$$v_* \to \omega* \qquad (2.4)$$

$$v \to \omega$$

For linear dimensions

$$d \to \delta \qquad (2.5)$$

$$k \to \kappa$$

Considering the hydraulic gradient I of a stream, this can be expressed for a pipe by the well-known Darcy-Weisbach equation. Writing the expression at the left in dimensional symbols and at the right in non-dimensional ones, one gets:

$$I = \frac{f}{2}\frac{V^2}{gD} = \frac{4v_*^{\,2}}{gD} \to I = \frac{f}{2}\frac{\Omega^2}{\Delta^\circ} = \frac{4\omega_*^{\,2}}{\Delta^\circ} \qquad (2.6)$$

In the present paper one will deal with the following stream boundaries:

a) A smooth boundary, investigated by Prandtl [9].

b) A boundary of non-uniform roughness investigated by Colebrook and White [1].

Boundaries of uniform roughness are often associated in open channel hydraulics with particle beds of constant size. Experiments by Shields [11] did define for these deformable boundaries conditions of incipient motion of a particle by a critical drag force.

Defining by $\frac{\rho_s}{\rho}$ the ratio between the solid and the fluid densities and expressing the above ratio in function of an excess density μ, let one write:

$$\frac{\rho_s}{\rho} = 1 + \mu \qquad (2.7)$$

In Shields' experimental work the non-dimensional group

$$Sh = \frac{1}{\mu}\,\frac{4v_*^2}{gd} \rightarrow Sh = \frac{1}{\mu}\,\frac{4\omega_*^2}{\delta} \qquad (2.8)^{\#}$$

has been represented versus the Reynolds number Re* of the particle at the boundary, expressed by:

$$Re^* = \frac{v^* d}{\nu} \rightarrow Re^* = \omega^*\delta \qquad (2.9)$$

A well-known functional correspondence in the form of a narrow band containing all the experimental points could be so defined by Shields. He found that for Re* = 1000 experimentation could not be carried on properly as the particle bed became unstable, i.e. particles show a tendency to leave the boundary for the stream. At Re* = 1000 Shields' functional values resulted to be inside the interval:

$$0{,}0551 < Sh < 0{,}0607 \qquad (2.10)$$

The present work will be closely related to this condition as discussed later.

Beside the definition of the functional Sh as per Eq. (2.8) one will make use of an other functional SH of expression:

$$SH = \frac{1}{\mu}\,\frac{f}{2}\,\frac{V^2}{gD} = \frac{1}{\mu}\,\frac{4v_*^2}{gD} \rightarrow SH = \frac{1}{\mu}\,\frac{f}{2}\,\frac{\Omega^2}{\Delta^\circ} = \frac{1}{\mu}\,\frac{4\omega_*^2}{\Delta^\circ} \qquad (2.11)$$

where SH is referred to the pipe diameter by substitution in Eq. (2.8) of $d \rightarrow \delta$ with $D \rightarrow \Delta^\circ$.

The following subscripts will be used in the context in order to denote a quantity associated with various conditions existing inside the stream.

Considering a stream over a particle bed in a flume of depth H, with particles of size d_* as in Shields' experiments, one can pass to an equivalent stream flow in a pipe of diameter D = 4H, with particles of size $d = 4d_*$ as given by Eq. (2.8).

Denoting by x the volumetric concentration of the solid particles transported let one define an “indifferent” particle one of the same density as the fluid. Hence for $1+\mu \rightarrow 1$ i.e. for $\mu \rightarrow \mu_{10} \rightarrow 0$, being μ_{10} a very small excess density one will use the subscript 1 for:

$$1 + \mu \rightarrow 1 \qquad \text{(a)}$$

Further for one solid particle only present in the system (of rather small diameter) one puts for the concentration

$$x \rightarrow 0 \qquad \text{(b)}$$

Hence combining the two conditions in subscript one writes:

$$\begin{aligned} &V_{10} \rightarrow \Omega_{10} \\ &D_{10} \rightarrow \Delta^{\circ}_{10} \\ &d_{10} \rightarrow \delta_{10} \\ &I_{10} \quad - \end{aligned} \qquad (2.12)$$

Further for $1 + \mu \rightarrow 1$ and $x > 0$ one writes:

$$\begin{aligned} &V_{lx} \rightarrow \Omega_{lx} \\ &D_{lx} \rightarrow \Delta^{\circ}_{lx} \\ &d_{lx} \rightarrow \delta_{lx} \\ &I_{lx} \quad - \end{aligned} \qquad (2.13)$$

For $\mu \neq 0$ but $x \rightarrow 0$, i.e. for one small particle of excess density μ present in the system let one put:

$$\begin{aligned} &V_{\mu o} \rightarrow \Omega_{\mu o} \\ &D_{\mu o} \rightarrow \Delta^{\circ}_{\mu o} \\ &d_{\mu o} \rightarrow \delta_{\mu o} \\ &I_{\mu o} \quad - \end{aligned} \qquad (2.14)$$

Finally for the concentration $x > 0$ and excess density $\mu \neq 0$ one writes:

$$\begin{aligned} &V_{\mu x} \rightarrow \Omega_{\mu x} \\ &D_{\mu x} \rightarrow \Delta^{\circ}_{\mu x} \\ &d_{\mu x} \rightarrow \delta_{\mu x} \\ &I_{\mu x} \quad - \end{aligned} \qquad (2.15)$$

In the analytical discussion of the experimental results retrieved from the technical literature (all dealing with water streams) in absence of a water temperature associated with a certain test, the author has assumed 19°C with a corresponding kinematic viscosity for water equal to $1\text{x}10^{-6}\left(\frac{m^2}{s}\right)$. Hence the numerical values of the transformation constants (2.2) and (2.3) are for water as below:

$$(g\nu)^{\frac{1}{3}} = 2{,}14\text{x}10^{-2}\left(\frac{m}{s}\right) \tag{2.16}$$

$$\left(\frac{g}{\nu^2}\right)^{\frac{1}{3}} = 2{,}14\text{x}10^{4}\left(\frac{1}{m}\right) \tag{2.17}$$

As well-known, values of friction factors are available in the technical literature in graphical forms.

In case (a) the determination has been carried out by Prandtl in case (b) calculations are due to Moody [7].

III. CONDITIONS DEFINING PARTICLE SUSPENSION

The condition of a particle transported in full suspension is expressed by the following relationship:

$$SH_{\mu o} = \frac{I_{\mu o}}{\mu} = 0{,}0607 \qquad (3.1)$$

i.e.

$$I_{\mu o} = 0{,}0607\mu \qquad (3.2)$$

with Eqs. (3.1) and (3.2) related to $\mu >< 0$.. The case $\mu < 0$ will be associated to a particle buoyant in the fluid.

Eq. (3.1) written in explicit form results as below:

$$SH_{\mu 0} = \frac{1}{2}\frac{f_{\mu o}}{\mu}\frac{\Omega^2_{\mu o}}{\Delta^\circ_{\mu o}} = 0{,}0607 \qquad (3.3)$$

Considering a particle of diameter $\delta_{\mu o}$ the above expression (3.3) can be written in the form:

$$Sh_{\mu 0} = \frac{1}{2}\frac{f_{\mu o}}{\mu}\frac{\Omega^2_{\mu o}}{\delta_{\mu o}} = 0{,}0607\frac{\Delta^\circ_{\mu o}}{\delta_{\mu o}} \qquad (3.4)$$

Eq. (3.4) for $\frac{1}{2}\frac{f_{\mu o}}{\mu}\frac{\Omega^2_{\mu o}}{\Delta_{\mu o}}$ = constant is just a linear relationship of $\delta_{\mu o}$ against itself.

For a friction velocity:

$$\left(\frac{f_{\mu o}}{8}\right)^{\frac{1}{2}} \Omega_{\mu o} = \omega_{*\mu o} = \text{constant} \qquad (3.5)$$

one defines the field of variation of $\delta_{\mu o}$ between the limits:

$\delta_{\mu o} = \Delta^{\circ}_{\mu o}$ and $\delta'_{\mu o}$ with $\delta'_{\mu o}$ defined, e.g. for a smooth pipe, by the relation:

$$\delta'_{\mu o} \times \omega_{*\mu o} = 10 \qquad (3.6)^{\#}$$

i.e. by

$$\delta'_{\mu o} = \frac{10}{\omega_{*\mu o}} \qquad (3.7)$$

Hence Eq. (3.4) will be considered inside the field of variation of the ratio $\frac{\Delta^{\circ}_{\mu o}}{\delta_{\mu o}}$ as given below:

$$1 < \frac{\Delta^{\circ}_{\mu o}}{\delta_{\mu o}} < \frac{\Delta^{\circ}_{\mu o}}{\delta'_{\mu o}} = 10^{4,059} = 11\ 455 \qquad (3.8)$$

being

$$4{,}059 = 16{,}47^{\frac{1}{2}} = \left(\frac{1}{0{,}0607}\right)^{\frac{1}{2}} = \ell g\ 11455 = \ell g\ \frac{\Delta^{\circ}_{\mu o}}{\delta'_{\mu o}} \qquad (3.9)$$

Hence with Eq. (3.4) as an experimental datum, and for a particle inside the interval (3.8), this equation insures full suspension of the particle by the stream.

In Fig. 1 Eq. (3.4) has been represented by the segment AB in a $Sh_{\mu o}$ versus $\frac{\Delta_{\mu o}}{\delta_{\mu o}}$ linear relationship (continuous line). The locus AB intersects the axis of the ordinates at the value Y = 0,0607 for $X = \frac{\Delta^{\circ}_{\mu o}}{\delta_{\mu o}} =$

At the value $Re_{*\mu o} = \omega_{*\mu o}\ \delta_{*\mu o} = 2{,}50$ of the variable, the Sh function for a particle bed in a flume has increased to its original value 0,0607. Then, let one consider the flow in a pipe, where for $\delta'_{\mu o} = 4\delta_{*\mu o}$ one will get Eq. (3.6). Hence the Sh function for pipe flow will acouire the value 0,0607 at $R'e_{*\mu o} = 10$.

1 as expected. Hence the slope of the AB locus is 0,0607. Next to AB one has drawn (in dotted line) the locus corresponding to a slope 0,0551 in agreement with the Sh experimental value 0,0551 as per expression (2.10). This value defines the width of the band of Shields experiments on the lower side.

Let one introduce now the condition:

$$\mu \to \mu_{10} \to 0 \tag{3.10}$$

by which the particle becomes "indifferent" to the force due to gravity. Let one write in correspondence of Eq (3.10):

$$f_{\mu o} \to f_{10},\ I_{\mu o} \to I_{10}\ ;\ \Omega_{\mu o} \to \Omega_{10} \text{ and } \Delta^{\circ}_{\mu o} \to \Delta^{\circ}_{10} \tag{3.11}$$

In the representation Ω_{10} versus Δ°_{10} the corresponding plane is herein denoted as "generating plane". Let one write the above equations referred to the generating plane with the same reference number but with a star sign as shown below:

$$SH^{\circ}_{10} = \frac{I_{10}}{\mu_{10}} = 0{,}0607 \tag{3.1.*}$$

$$I_{10} = 0{,}0607\ \mu_{10} \tag{3.2.*}$$

$$SH^{\circ}_{10} = \frac{1}{2}\frac{f_{10}}{\mu_{10}}\frac{\Omega^2_{10}}{\Delta^{\circ}_{10}} = 0{,}0607 = \frac{1}{4{,}059^2} \tag{3.3.*}$$

$$Sh_{10} = \frac{1}{2}\frac{f_{10}}{\mu_{10}}\frac{\Omega^2_{10}}{\delta_{10}} = 0{,}0607\ \frac{\Delta^{\circ}_{10}}{\delta_{10}} \tag{3.4.*}$$

$$\left(\frac{f_{10}}{8}\right)^{\frac{1}{2}} x\ \Omega_{10} = \omega_{*10} \tag{3.5.*}$$

$$\delta'_{10}\ x\ \omega_{*10} = 10 \tag{3.6.*}$$

$$\delta'_{10} = \frac{10}{\omega_{*10}} \tag{3.7.*}$$

$$1 < \frac{\Delta^{\circ}_{10}}{\delta_{10}} < \frac{\Delta^{\circ}_{10}}{\delta'_{10}} = 10^{4,059} = 11455 \qquad (3.8.*)$$

With gravitational forces playing a marginal role inside the generating plane, only forces of inertia acting upon the particle ought to be considered.

Let one introduce in the generating plane the following reference condition:

$$\Delta^{\circ}_{10} = \Omega^{2}_{10} \qquad 3.12)$$

Thus with Eq. (3.12) into (3.3.*) one gets:

$$\frac{1}{2}\,\frac{f_{10}}{\mu_{10}} = 0{,}0607 \qquad (3.13)$$

IV. MINIMUM ENERGY FUNCTIONS DEFINED IN THE EXPERIMENTAL FIELD

From the analysis (by elementary methods) of a minimum energy dissipative function, certain simple relations have been obtained by the author as discussed below:

Omitting the rather long analytical derivation they are introduced as "acceptable" experimental expressions.

(a) In the generating plane this relation is simply given by

$$\Omega_{10} \text{ x } \Delta^{\circ}_{10} = RE_{10} = \text{constant} \tag{4.1}$$

being RE_{10} in Eq. (4.1) the Reynolds number of the stream in the generating plane.

The relation (4.1) is to be interpreted as follows:

For a particle in equilibrium under a force of inertia and a hydraulic force in vicinity of the boundary, the equilibrium is maintained along the hyperbola RE_{10} = constant.

(b) outside the generating plane let one consider the conditions $\mu \neq 0$ and $x \to 0$, i.e. one deals with a single small particle of excess density μ, which is present in the system.

Hence one writes (from the theory) the condition of inertial equilibrium for the particle as follows:

$$\Omega_{\mu o} \text{ x } \Delta^{\circ}_{\mu o} = RE_{10}\,(1 + \mu) = RE_{\mu o} \tag{4.2}$$

with $RE_{\mu o}$ defined by Eq. (4.2).

Introducing the exponents α and $1\text{-}\alpha$ one puts

$$\Omega_{\mu o} = \Omega_{10} (1 + \mu)^{\alpha} \tag{4.3}$$

$$\Delta^{\circ}_{\mu o} = \Delta^{\circ}_{10} (1 + \mu)^{1-\alpha} \tag{4.4}$$

Hence from Eqs. (4.3), (4.4) into (4.2) one gets with (4.1):

$$(1 + \mu)^{\alpha} (1 + \mu)^{1-\alpha} = 1 + \mu \tag{4.5}$$

Eq. (4.5) is the first equilibrium equation derived from the minimum energy functional principle applied outside the generating plane.

(c) Equally outside the generating plane for a concentration $x > 0$ but for a particle of excess density $\mu_{10} \to 0$, i.e. for a particle of size δ_{lx} let one denote by ε an increment in stream velocity. Then one puts:

$$\Omega_{lx} = \Omega_{10} (1 + \varepsilon) \tag{4.6}$$

$$\Delta^{\circ}_{lx} = \Delta^{\circ}_{10} (1 - x) \tag{4.7}$$

Eqs. (4.6) and (4.7) express the following condition:

To an increase in stream velocity to $1 + \varepsilon$ there correspond a decrease in stream size to $1 - x$.

Hence by multiplication of (4.6) and (4.7) one obtains on account of (4.1)

$$(1 + \varepsilon) (1 - x) = 1 \tag{4.8}$$

Eq. (4.8) is the second equilibrium equation derived from the minimum energy functional principle, outside the generating plane.

V. COMBINED EQUILIBRIUM EQUATIONS

(b) Outside the generating plane one will consider, with $\mu \neq 0$ and $x \to 0$, i.e. for a single particle of excess density μ, the system of equations (3.3) and (4.5).

Further with relations (4.3), (4.4) and (3.12) one obtains:

$$\frac{1}{2}\,\frac{f_{\mu o}}{\mu}\,\frac{(1+\mu)^{2\alpha}}{(1+\mu)^{1-\alpha}} = 0{,}0607 \qquad (5.1)$$

From Eq. (5.1) one obtains:

$$f_{\mu o} = 0{,}1214\mu\;(1+\mu)^{1-3\alpha} \qquad (5.2)$$

Division of $\Omega^2_{\mu o}$ by $\Delta^\circ_{\mu o}$ yields on account of Eqs. (4.3), (4.4) and (3.12):

$$\frac{\Omega^2_{\mu o}}{\Delta^\circ_{\mu o}} = \frac{(1+\mu)^{2\alpha}}{(1+\mu)^{1-\alpha}} = (1+\mu)^{3\alpha-1} \qquad (5.3)$$

i.e.

$$\alpha = \left\{\frac{\ell g\,\dfrac{\Delta^\circ_{\mu o}}{\Omega_{\mu o}{}^2}}{\ell g\,(1+\mu)} - 1\right\} \Big/ (-)\;3{,}0 \qquad (5.4)$$

(c) Outside the generating plane let one consider with $\mu \to \mu_{10} \to 0$, a certain volumetric concentration $x > 0$.

Referring at first to a particle having the form of a cylindrical body of diameter δ_{lx}, let one consider δ_{lx} capable of approaching the diameter of the pipe Δ°_{10}.

Let one express the volumetric concentration for such a particle by the ratio square as below:

$$x = \left(\frac{\delta_{lx}}{\Delta^{\circ}_{10}} \right)^2 \tag{5.5}$$

Further one introduces the following assumption:

that the fluid stream and the particle proceed at the same nominal velocity of the mixture, i.e. no slippage between the solid and the liquid phases can occur. This assumption can be verified on a system of fully suspended particles by an exchange of momentum between particles and fluid and vice versa, resulting in a perfect mixing of the two phases. From this one infers that such a system allows the definition of the solid concentration as per Eq. (5.5) both from the Lagrangian and from the Eulerian point of view, as each other identical.

Hence from Eq. (5.5) one gets for the definition of the particle size in the generating plane:

$$\lim_{x \to 0} \delta_{lx} \to \delta_{10} = \delta'_{10} = \frac{\Delta^{\circ}_{10}}{11455} \tag{5.6}$$

(cf. Eq. 3.8)

From Eq (4.8) one gets:

$$\varepsilon = \frac{x}{1 - x} \tag{5.7}$$

Eq. (5.7) defines ε by its term at the right as the “relative” volumetric concentration.

By means of Eqs. (4.6), (4.7) and (4.8) let one write an expression analogous to Eq. (5.3). Thus taking into account condition (3.12) one gets:

$$\frac{\Omega_{l\varepsilon}^{2}}{\Delta_{lx}^{\circ}} = \frac{\Omega_{10}^{2}\,(1 + \varepsilon)^{2}}{\Delta_{10}^{\circ}\,(1 - x)} = (1 + \varepsilon)^{3} = \frac{1}{(1 - x)^{3}} \tag{5.8}$$

with

$$\Omega_{l\varepsilon} = \Omega_{10}\,(1 + \varepsilon)^{\frac{3}{2}} = \Omega_{10}\,\frac{1}{(1 - x)^{\frac{3}{2}}} \tag{5.9}$$

and

$$\Delta_{lx}^{\circ} = (1 - x)\,\Delta_{10}^{\circ} \tag{5.10}$$

Hence addition of x particles to the system has produced a decrease in the actual linear section available for transport as per Eq. (5.10).

Let one consider the case $x \to 1$, i.e. that of a cylindrical particle approaching the dimension of the pipe.

Hence one can define the volumetric concentration x instead of expression (5.5), by a linear concentration $\bar{x}$ as given below:

$$\bar{x} = \frac{\delta_{lx}}{\Delta_{10}^{\circ}} = x^{\frac{1}{2}} \tag{5.11}$$

Let one write Eq. (5.9) with $\bar{\varepsilon}$ and $\bar{x}$ in place of ε and x as below:

$$\Omega_{l\bar{\varepsilon}} = \Omega_{10}\,(1 + \bar{\varepsilon})^{\frac{3}{2}} = \Omega_{10}\,\frac{1}{(1 - \bar{x})^{\frac{3}{2}}} \tag{5.12}$$

being now

$$\bar{\varepsilon} = \frac{x^{\frac{1}{2}}}{1 - x^{\frac{1}{2}}} = \frac{cx}{1 - cx} = \frac{\bar{x}}{1 - \bar{x}} \tag{5.13}$$

In Eq. (5.13) one has purposely introduced a parameter c of expression

$$c = \frac{1}{x^{\frac{1}{2}}} \tag{5.14}$$

Eq. (5.14) is to be warranted by the following analytical development.

With c in the role of a suitable undefined multiplier of x let one write instead of (5.7):

$$\bar{\varepsilon} = \frac{\bar{x}}{1 - \bar{x}} = \frac{cx}{1 - cx} \qquad (5.15)$$

and from (5.12):

$$1 + \bar{\varepsilon} = \frac{1}{1 - \bar{x}} = \frac{1}{1 - cx} \qquad (5.16)$$

On account of the arbitrariness of c let one write

$$\frac{1}{1 - \bar{x}} = \frac{1}{1 - cx} = \frac{1 + cx}{1 - x} \qquad (5.17)$$

i.e.

$$1 - x = 1 - (cx)^2 \qquad (5.18)$$

i.e.

$$c = \frac{1}{x^{\frac{1}{2}}} \qquad (5.14)(\text{Repeated})$$

as expected.

With (5.13) into (5.12) one gets for the velocity $\Omega_{1\bar{\varepsilon}}$:

$$\Omega_{1\bar{\varepsilon}} = \Omega_{l\bar{x}} = \frac{\Omega_{10}}{(1 + x^{\frac{1}{2}})^{\frac{3}{2}}} \qquad (5.19)$$

In order to simplify notation let one put from now onward

$\Omega_{l\bar{x}} = \Omega_{lx}$ and write:

For the stream velocity

$$\frac{\Omega_{lx}}{\Omega_{10}} = \frac{1}{(1 - x^{\frac{1}{2}})^{\frac{3}{2}}} \tag{5.20}$$

For the hydraulic gradient I_{lx}

$$\frac{I_{lx}}{I_{10}} = \frac{1}{(1 - x^{\frac{1}{2}})^{3}} \tag{5.21}$$

The graphical representation of $(1 - x^{\frac{1}{2}})^{\frac{3}{2}}$ versus x is shown in Fig. 2 by the locus I.

This locus has its origin at the point {0, 1} on the axis of the ordinates and its end on the axis of the abscissae at point {1, 0}.

At these two points the geometrical tangent to the locus coincides with the corresponding axis.

In Fig. 2 the locus II represents the velocity ratio $\frac{\Omega_{lx}}{\Omega_{10}}$ as per Eq. (5.20), the locus III represents the hydraulic gradient ratio $\frac{I_{lx}}{I_{10}}$ as per Eq. (5.21).

The analysis so far developed has been referred to a cylindrical particle of diameter δ. Let one extend below the obtained results to a spherical particle of the same diameter δ.

One gets from the volumetric ratio between the two particles

$\frac{\frac{\pi\delta^3}{4}}{\frac{\pi\delta^3}{6}} = \frac{3}{2}$. Hence a spherical particle of the size of the pipe will occupy a volume $\frac{2}{3}$ = 0,6666 of the cylindrical particle of the same diameter.

Referring this ratio to the concentration $x^{\frac{1}{2}}$ one obtains for $x \to 1$ and $0,816 = \left(\frac{2}{3}\right)^{\frac{1}{2}}$

$$x^{\frac{1}{2}}_{sphere} = 0,816\ x^{\frac{1}{2}}_{cylinder}. \tag{5.22}$$

In Fig 3 one has represented the ratio $x^{\frac{1}{2}} = \frac{\delta}{\Delta^{\circ}_{10}}$ of a cylindrical particle and the ratio $x^{\frac{1}{2}} = \frac{\delta}{\Delta^{\circ}_{10}}$ of a spherical particle, this for any value of $0 < x < 1$. The diagram for the spherical particle starts for $\frac{\delta}{\Delta^{\circ}_{10}} = 1$ at the point of abscissa $x = 0{,}816$.

From Fig 3 one can infer the following:

Given a certain volumetric concentration x on the axis of the abscissae, referred to a cylindrical particle, the square root volumetric concentration $x^{\frac{1}{2}}$ of a spherical particle to reckon with, is 1,11 times greater than the square root concentration $x^{\frac{1}{2}}$ of a cylindrical particle. Hence when dealing with spherical particles one has to consider volumetric concentrations as below:

$$\beta x^{\frac{1}{2}}_{sphere} = x^{\frac{1}{2}}_{cylinder} = 1{,}11 \times x^{\frac{1}{2}}_{sphere} \text{ with } \beta = 1{,}11 \qquad (5.23)$$

In Fig. 3 one has shown in Insert (a) four particles in a compact assembly inside a cylindrical body of diameter δ and length 4δ. In Insert (b) the particles have lost their alignment and proceed inside the pipe under the influence of the stream:

VI. APPLICATIONS

Let one apply to the below development the conditions $x > 0$ and $\mu \neq 0$, sometimes separately, sometimes superimposed (cf. Ch V at b and at c).

From the energetic point of view scalar superimposition of effects is to be considered as physically legitimate.

Thus one deals with symbols of the form $\Omega_{\mu x}$ $\Delta^{\circ}_{\mu x}$ $\delta_{\mu x}$ $I_{\mu x}$ ...etc. Further in the μ, x plane the particle diameter $\delta_{\mu x}$ has a direct role in defining the volumetric concentration x as per Eqs. (5.5), (5.11) and (5.23).

1°) In a first worked example one calculates in the generating plane the hydraulic gradient required for the conveyance of a cylindrical body inside a smooth wall pipe.

Putting for the size δ_{10} of the cylindrical body the condition $\delta_{10} \rightarrow \Delta^{\circ}_{10}$ i.e. for $\delta_{10} \rightarrow \Delta^{\circ}_{10} - \eta$ being η a small quantity, it results from the calculations that a finite hydraulic gradient $I_{1-\eta}$ is required, of an order of magnitude, which can be verified experimentally.

Denoting by I_{lx} the hydraulic gradient one gets from Eq. (5.21) and (3.2*)

$$I_{lx} = \frac{I_{10}}{(1 - x^{\frac{1}{2}})^3} = \frac{0{,}0607\ \mu_{10}}{(1 - x^{\frac{1}{2}})^3} \tag{6.1}$$

For $x \rightarrow 1 - \eta$ one writes:

$$1 - x^{\frac{1}{2}} = 1 - (1 - \eta)^{\frac{1}{2}} \tag{6.2}$$

Developing $(1 - \eta)^{\frac{1}{2}}$ in a power series expansion, one writes for $(1 - x^{\frac{1}{2}})^3$ while neglecting terms of power greater than one:

$$\left\{1-(1-\eta)^{\frac{1}{2}}\right\}^3 = \left\{1-\left(1-\frac{\eta}{2}+\frac{1}{2}\,\frac{3}{2}\,\frac{\eta^2}{2}+\ldots\ldots\right)\right\}^3 \approx \frac{3\eta}{2} \qquad (6.3)$$

With the denominator of Eq. (6.1) expressed by (6.3) and putting because of the arbitrariness of η and of μ_{10}, (both small quantities):

$$\frac{\mu_{10}}{\eta} = 1 \qquad (6.4)$$

one obtains for a cylinder (of length $\Lambda = \Delta^{\circ}_{10}$)

$$I_{1,1-\eta} = \frac{2}{3} \times 0{,}0607 = 0{,}0405 \qquad (6.5)$$

which is the finite value of the hydraulic gradient insuring transport in absence of inertial impacts upon the walls.

From the technical literature dealing with capsule transport it is of interest to mention the work of Ellis [4]. In Fig. 2 of the referred paper the hydraulic gradient required to support a capsule of size $\frac{\delta_{\mu o}}{\Delta^{\circ}_{\mu o}} = 0{,}95$ is equal to $I_{\mu x} = 0{,}083$, this for a capsule of relative density $\frac{\rho_s}{\rho} = 1 + \mu = 1{,}033$.[#]

2°) In this second worked example one departs from conditions Ω_{10} and Δ_{10} defined in the generating plane and calculate $\Omega_{\mu o}$, $\Delta^{\circ}_{\mu o}$, $\Omega_{\mu x}$ and $I_{\mu x}$ for a certain particle of size $\delta_{\mu x}$.

This will be applied to the transport of sand in an air stream.

The data of the problem are given as below:

Air temperature 30°C, kinematics viscosity of the air 16 x $10^{-6}\left(\frac{m^2}{s}\right)$.

For the velocity scale (cf. Eq. 2.2):

[#] Applying to $I_{\mu x} = 0.083$ the values of the experiments $\eta = 1 - 0{,}95 = 0{,}050$ and $\mu = 0{,}033$ one gets for the calculated value $I_{\mu x\ calc} = 0{,}083 \times \frac{0{,}033}{0{,}050} = 0{,}0548$

$(g\nu)^{\frac{1}{3}} = 5{,}391 \text{ x } 10^{-2} (\frac{m}{s})$

For the distance scale (cf. Eq. 2.3)

$\left(\frac{g}{\nu^2}\right)^{\frac{1}{3}} = 3370 \left(\frac{1}{m}\right)$

For a sand/air density ratio $\frac{\rho_s}{\rho} = 1 + \mu = 2285$

For a water/air density ratio $\frac{\rho_w}{\rho} = 1 + \mu' = 862{,}3$

one gets from Eq. (5.2):

$f_{\mu o} = 0{,}1214\mu\,(1+\mu)^{1-3\alpha}$

Select for α the value 0,730 i.e.

$1 - 3\alpha = 1 - 3 \text{ x } 0{,}730 = -1{,}19$

Hence from Eq. (5.2).

$f_{\mu o} = 0{,}1214 \text{ x } 2284 \text{ x } 2285^{-1{,}19} = 0{,}02791.$

Let one consider an unspecified small particle and a smooth wall pipe of diameter Δ°_{10} as below:

$\Delta^{\circ}_{10} = 3{,}370.$

Further on account of Eq. (3.12).

$\Omega_{10} = \Delta^{\circ\frac{1}{2}}_{10} = 1{,}836.$

Then from Eqs. (4.3) and (4.4)

$\Omega_{\mu o} = \Omega_{10}\,(1 + \mu)^{\alpha} = 1{,}836\text{x}2285^{0{,}730} = 519{,}83$

$\Delta^{\circ}_{\mu o} = \Delta^{\circ}_{10}(1 + \mu)^{1-\alpha} = 3{,}370\text{x}2285^{0{,}270} = 27{,}20$

Expressing the Reynolds number of the stream $RE_{\mu o}$, this is given by:

$RE_{\mu o} = \Delta^{\circ}_{\mu o} \text{ x } \Omega_{\mu o} = 519{,}8\text{x}27{,}20 = 14139$:

At the above Reynolds number the value $f_{\mu o}$ = 0,02791 is in agreement with the value selected.

Hence one can write as per Eq. (2.6)

$$I_{\mu o} = \frac{f_{\mu o}}{2} \text{ x } \frac{\Omega^2_{\mu o}}{\Delta^\circ_{\mu o}} = \frac{0,02791}{2} \frac{519,8^2}{27,20} = 138,64$$

and

$$SH_{\mu o} = \frac{I_{\mu o}}{\mu} = \frac{138,64}{2284} = 0,0607$$

Let assume that the very small particle has been increased to a reasonable size, e.g. to 1 mm i.e. with $\delta_{\mu x}$ = 3,37 the stream carries a line of cylindrical particles of diameter ratio

$$\frac{\delta_{\mu x}}{\Delta^\circ_{\mu o}} = \frac{3,37}{27,20} = 0,1237 = x^{\frac{1}{2}}$$

as per Eq. (5.11).

Further taking into account the factor β = 1,11 as per Eq. (5.23) one gets from Eq. (5.20) applied to spherical particle, i.e. for $\beta x^{\frac{1}{2}}$ = 1,11x0,1237 = 0,137

$$\Omega_{\mu\beta x} = \frac{\Omega_{\mu o}}{(1 - \beta x^{\frac{1}{2}})^{\frac{3}{2}}} = \frac{519,8}{(1 - 0,137)^{\frac{3}{2}}} = 648,4$$

Analogously for the hydraulic gradient

$$I_{\mu\beta x} = \frac{I^\circ_{\mu o}}{(1 - \beta x^{\frac{1}{2}})^3} = \frac{138,64}{(1 - 0,137)^3} = 215,7$$

Passing to dimensional symbols by the scale factors introduced above one gets for the dimensional velocity and pipe diameter

$$V_{\mu x} = \Omega_{\mu x} \text{ x } 5,391 \text{ x } 10^{-2} = 648,4 \text{ x } 5,391 \text{ x } 10^{-2} = 34,95 \text{ (m/s)}$$

$$D_{\mu x} = \frac{\Delta^{\circ}_{\mu x}}{3370} = \frac{27{,}20}{3370} = 0{,}00807 \text{ (m) i.e. } 8{,}07 \text{ (mm)}$$

and

$$d_{\mu x} = \frac{\delta_{\mu x}}{3370} = \frac{3{,}370}{3370} = 0{,}001 \text{ (m) i.e. } 1 \text{ (mm)}$$

Further expressing the hydraulic gradient $I_{\mu x}$ in meters of water column per meter of pipe, i.e. dividing by 862,3 one gets

$$I_{\mu x} = \frac{215{,}7}{862{,}3} = 0{,}250 \left(\frac{m}{m}\right)$$

Finally for $x^{\frac{1}{2}} = 0{,}1237$ one gets $x = 0{,}0153$ as the actual volumetric concentration of sand particles transported.

3°) As a third worked example one calculates the suspension velocity $\Omega_{\mu o}$ for a buoyant particle of excess density $\mu = -0{,}50$ i.e. for $1 + \mu = 1 - 0{,}50 = 0{,}50$

Let one consider a flow in a smooth pipe

For a stream Reynolds number

$$RE_{\mu o} = \Omega_{\mu o} \text{ x } \Delta^{\circ}_{\mu o} = 30000$$

one gets a friction factor $f_{\mu\, o\ smooth} = 0{,}0240$.

Let one introduce the convention that to a buoyant particle a negative friction factor $f_{\mu o}$ applies. Hence writing in Eq. (5.2) $f_{\mu o} = -0{,}0240$ one gets:

$$\alpha = \left\{ 1 - \frac{\ell g \dfrac{f_{\mu o}}{0{,}1214\mu}}{\ell g\ (1+\mu)} \right\} \Big/ 3{,}0 \text{ i.e. } \alpha = \left\{ 1 - \frac{\ell g \dfrac{-0{,}0240}{-0{,}1214 \text{ x } 0{,}50}}{\ell g\ 0{,}50} \right\} /3{,}0 = -0{,}113$$

with $\alpha = -0{,}113$ substituted in Eqs. (4.3) and (4.4) one gets with (3.12):

$$\frac{\Omega_{\mu o}^{2}}{\Delta_{\mu o}^{\circ}} = \frac{(1+\mu)^{2\alpha}}{(1+\mu)^{1-\alpha}} = (1+\mu)^{3\alpha-1} = 0{,}50^{-1{,}339} = 2{,}530 \qquad \text{(I)}$$

Further

$$\Omega_{\mu o} \times \Delta_{\mu o}^{\circ} = RE_{\mu o} = 30000 \qquad \text{(II)}$$

Hence from (I) and (II):

$\Omega_{\mu o} = 42{,}32$

$\Delta_{\mu o}^{\circ} = 708{,}8$

and from Eqs. (4.3) and (4.4) one gets in the generating plane:

$$\Omega_{10} = \frac{\Omega_{\mu o}}{(1+\mu)^{\alpha}} = \frac{42{,}32}{0{,}500^{-0{,}113}} = 39{,}13$$

and

$$\Delta_{10}^{\circ} = \frac{\Delta_{\mu o}^{\circ}}{(1+\mu)^{1-\alpha}} = \frac{708{,}8}{0{,}500^{1{,}113}} = 1533$$

with values of Ω_{10} and of Δ_{10}° satisfying Eq. (3.12) as it should be.

VII. COMPARISON OF EXPERIMENTAL RESULTS WITH THOSE OBTAINED FROM THE THEORY

Sixteen cases of hydraulic transport are to be discussed and presented in their main operative parameters as per Table 1. Column 1 of Table 1 carries an internal reference number. Column 2 carries the literature reference number of the publication from which information was extracted and in brackets the reference figures. Column 3 carries the solid-water density ratio $\frac{\rho_s}{\rho} = 1 + \mu$, Column 4 the pipe diameter D, Column 5 the particle diameter d, Column 6 the relative roughness of the pipe $\frac{k}{D}$, Column 7 the friction factor $f_{\mu o}$, Column 8 the suspension velocity $V_{\mu o}$ and Column 9 the hydraulic gradient $I_{\mu o}$ associated with $V_{\mu o}$ (cf Eq. 3.2). In Table 1 the solid concentration does not appear as one refers to one small particle only, present.

In Table 2 quantities are expressed in a non-dimensional form as per Eqs. (2.16) and (2.17).

Columns 1, 2 and 6 of Table 2 are just a repetition of the corresponding columns 1, 3 and 7 of Table 1. Column 3 and 5 carry non-dimensional values of the pipe diameter $D_{\mu o}$ and of the velocity $V_{\mu o}$ of Table 1. Calculation of α by Eq. (5.4) has enabled the definition of Ω_{10} and Δ°_{10} in the generating plane. In Table 3 the tabulated quantities refer to a field with $\mu \neq 0$ and $x > 0$. With Column 1 used as reference, the quantities $1 + \mu$, $\Delta^{\circ}_{\mu o}$, $\delta_{\mu o}$ and $\Omega_{\mu o}$ in Columns 2, 3, 4 and 5 are those of the corresponding Columns 2, 3 4 and 5 of Table 2. Column 6 carries the hydraulic gradient $I_{\mu o}$ (already reported in Table 2 by Column 7). Column 7, 8 and 9 carry the concentration x, $x^{\frac{1}{2}}$ and $\beta x^{\frac{1}{2}}$ in its various forms. Columns 10 and 11 carry the stream velocity $\Omega_{\mu\beta x}$ and the hydraulic gradient $I_{\mu\beta x}$, referred to conditions $\mu\,\beta x$ respectively. Column 12 carries the experimental hydraulic gradient $I_{\mu\,x\ \text{experim}}$. measured graphically in the plane of representation of the test results. Column 13 carries the ratio between the calculated values of $I_{\mu\beta x}$ and the experimental ones $I_{\mu\, x\ experim}$.

Many values of $\Omega_{\mu\beta x}$ and $I_{\mu\beta x}$ calculated according to the theory could not be analysed by comparison. Their experimental counterparts could not be ascertained, being the experimental field too narrow.

With reference to experiment 8 the original percental concentration by weight has been reduced by the author to a volumetric concentration.

Hence Table 3 covers twenty-four results verified experimentally and two results obtained by extrapolation.

As a graphical illustration of the method let one refers to experiment 5 and to the corresponding Fig. 4 (retraced by the author). The experiment deals with the hydraulic transport of gravel $\left(\frac{\rho_s}{\rho} = 1 + \mu = 2{,}55\right)$. Characteristical loci are shown in a representation hydraulic gradient I versus a nominal velocity V (solid + liquid):

A locus for water only and five loci of increasing volumetric concentrations x = 0,05; 0,10; 0,15; 0,20 and 0,25 have been drawn:

In Table 3 at reference 5 one has reported values of $\Omega_{\mu o}$ and $I_{\mu o}$ and then values $\Omega_{\mu\beta x}$ and $I_{\mu\beta x}$ with representative points falling close to the corresponding characteristical loci for x = 0,05; 0,10 and 015.

For the value x = 0,20 the representative point has been extrapolated as shown. Hence in Fig. 4 a locus DC (in dotted line) has been drawn joining all the points obtained by the theory.

The statistical analysis of the twenty-four numerical results of Column 14 (results by extrapolation excluded) yields a mean close to 1 (0,982) and a standard deviation $\sigma = 0{,}050$.

VIII. CONCLUSIONS

In Ch. I one has outlined the scope and the results of the investigation.

In Ch. II one has introduced non-dimensional quantities expressing a velocity and a length. This has made the ensuing analytical development independent of the kinematic viscosity and of the acceleration due to gravity. Hydraulic laws considered have been referred to pipe flow in presence of a smooth boundary or of a boundary of non-uniform roughness.

The hydraulic gradient of a stream I as per Eq. (2.6), the Shields function Sh, as per Eq. (2.8) (referred to a particle of size δ) and the Shields function SH as per (Eq. 2.11), (referred to a pipe diameter $\Delta°$) have been formally introduced and associated with the numerical values of Eq. (2.10).

In Ch. III critical values defining particle suspension have been expressed by means of relationships (3.1), (3.2) and more explicitly by (3.3) and (3.4).

In all these relationships with $\mu \neq 0$ the role of the particle size $\delta_{\mu o}$ resulted "irrelevant". For $\mu \to \mu_{10} \to 0$, i.e. with μ_{10} a small quantity, one has defined an "indifferent" particle and a "generating" plane. In this plane the force due to gravity plays a marginal role, so only the force of inertia of a particle remains conspicuous.

In Ch. IV one has introduced the concept of a minimum energy dissipative function to be (ultimately) associated to the hydraulic gradient I.

(a) In the generating plane this has been expressed simply by Eq. (4.1).

(b) Outside the generating plane for $\mu \neq 0$ and $x \to 0$ i.e. for a very small particle (cf. Eq. (3.8) for $\delta_{\mu o} \cong \delta_{\mu o}$) the associated equation is (4.5). Hence the condition $x \to 0$ introduced above has limited considerably the variational field (3.8) to $\delta_{\mu o} \cong \delta'_{\mu o} = \frac{\Delta^{\circ}_{\mu o}}{11455}$.

(c) Outside the generating plane for $\mu \to \mu_{10} \to 0$ and $x \neq 0$, relations (4.6) and (4.7) yield the second equation (4.8) insuring particle equilibrium against forces of inertia.

In Ch. V at (b) conditions of equilibrium under gravitational forces as per Eq. (3.4) have been combined with the conditions of equilibrium for inertial forces (4.3), (4.4) and (4.5) for $\mu \neq 0$ but $x \to 0$, i.e. for a particle of a small diameter $\delta_{\mu o} \approx \delta'_{\mu o}$.

In (c) one has departed from condition of equilibrium in the generating plane as per Eq. (4.1) and further discussed condition of equilibrium (4.8).

The analysis has required an insight definition of the volumetric concentration x. This was referred at first to a cylindrical body, alternatively to a spherical particle. The concentration x for a cylindrical body has been expressed by the square ratio of the diameters as per Eq. (5.5). A relative concentration has been defined by means of Eq. (5.7) and finally a linear concentration $\bar{x}$ expressed by the linear ratio of the diameters.

Analytical development led to an expression of the stream velocity Ω_{lx} as ratio $\frac{\Omega_{lx}}{\Omega_{10}}$ as per Eq. (5.20). Hence an expression of the corresponding hydraulic gradient I_{lx} as ratio $\frac{I_{lx}}{I_{10}}$ as per Eq (5.21)

Where dealing with a spherical particle the linear concentration of the particle has been obtained from that at the same diameter cylinder inscribing the particle, i.e. by multiplication by the factor β = 1,11, as given by Eq. (5.23).

In Ch. VI worked examples have been carried out by applying the literary expressions established in the analysis.

In Ch. VII experimental results have been discussed in a $\mu \neq 0$ and $x > 0$ field, with velocities and hydraulic gradients defined as below:

$$\frac{\Omega_{\mu o}}{(1 - \beta x^{\frac{1}{2}})^{\frac{3}{2}}} = \Omega_{\mu\beta x} \quad (8.1)$$

$$\frac{I_{\mu o}}{(1 - \beta x^{\frac{1}{2}})^{3}} = I_{\mu\beta x} \quad (8.2)$$

$$I_{\mu o} = \frac{f_{\mu o}}{2} \frac{\Omega_{\mu o}^{2}}{\Delta_{\mu o}^{\circ}} = 0{,}0607\ \mu \quad (8.3)$$

A remark: Fig 1 should be always referred to a very small solid concentration, e.g. to one particle only present in the system. This applies also to the case $\frac{d}{D} \rightarrow 1$, i.e. to spherical particle of the size of the pipe, for which the local concentration approaches the value 0,666.

TABLE 1: EXPERIMENTAL PARAMETERS RELATED TO SIXTEEN TEST RESULTS EXTRACTED FROM THE TECHNICAL LITERATURE DEFINING SUSPENSION VELOCITY OF A PARTICLE AND SUSPENSION HYDRAULIC GRADIENT.

1	2		3	4	5			6	7	8	9
REF	LITERATURE REFERENCE & (FIGURE NUMBER)		DENSITY RATIO $\frac{\rho_s}{\rho} = 1 + \mu$	PIPE DIA. $D_{\mu O}$ (mm)	PARTICLE DIA. (mm)			RELAT. ROUGHNESS $\frac{k}{D} \times 10^3$	FRICTION FACTOR $f_{\mu o}$	SUSPENSION VELOCITY $V_{\mu o}$ (m/s)	SUSPENSION HYDR GRAD. $I_{\mu o}$
					MIN	MEDIAN d	MAX				
1	14	(24)	1,40	38	9,0	12,5	15	1,30	0,0267	0,823	0,0243
2	15	(3.b)	1,40	150	Not given	Not given	38	0,40	0,0173	2,035	0,0243
3	15	(3.a)	1,40	76	Not given	12,7	19	0,20	0,0190	1,380	0,0243
4	6	(3)	1,38	25,4	–	3,93 one size	–	smooth	0,0270	0,6527	0,0231
5	8	(16)	2,55	25,4	2,50	3,75	5	smooth	0,0220	1,459	0,0941
6	16	(14.3)	2.61	80,7	6,0	8,0	10	0,60	0,0190	2,853	0,0977
7	10	(3)	11,30 LEAD	32,0	–	2,46 one size	–	smooth	0,0160	4,952	0,6255
8	12	(11)	2,93	25,4	1,19	1,38	1,41	smooth	0,0215	1,648	0,1172
9	5	(2)	2,65	25,4	0,50	0,57	1,00	0,50	0,0240	1,442	0,1001
10	8	(15)	2,65	25,4	Not given	0,75	Not given	0,20	0,0235	1,457	0,1001
11	2	(19)	2,65	150,0	1,62	2,04	2,46	0,030	0,0128	4,795	0,1001
12	8	(14)	2,65	25,4	0,13	0,21	0,25	0,055	0,0220	1,5055	0,1001
13	13	(2)	2,67	88,4	Not given	0,69	Not given	0,100	0,0155	3,366	0,1014
14	16	(14.1)	2,65	155,2	0,50	0,91	1,50	0,280	0,0156	4,419	0,1001
15	4	(3)	8,85 NICKEL	25,4	0,05	0,13	0,17	smooth	0,0180	3,632	0,4764
16	2	(18)	2,65	150,0	0,34	0,44	0,54	0,030	0,0129	4,776	0,1001

TABLE 2: DETERMINATION OF CONDITIONS IN THE GENERATING PLANE FROM CONDITIONS OF PARTICLE IN SUSPENSION I.E. $\Delta^{o}_{\mu o}$ and $\Omega_{\mu o}$

1	2	3	4	5	6	7	8	9	10	11	12
REF.	$1+\mu$	$\Delta^{o}_{\mu o}$	$\delta_{\mu o}$	$\Omega_{\mu o}$	$f_{\mu o}$	$\frac{f_{\mu o}}{2}\frac{\Omega^2_{\mu o}}{\Delta^{o}_{\mu o}} = I_{\mu o}$	$\left\{\frac{\ell g \frac{f_{\mu o}}{0,1214\mu}}{\ell g\,(1+\mu)} - 1\right\} \Big/ (-)3,0 = \alpha$	$(1+\mu)^{\alpha}$	$(1+\mu)^{1-\alpha}$	$\frac{\Omega_{\mu o}}{(1+\mu)^{\alpha}} = \Omega_{10}$	$\frac{\Delta^{o}_{\mu o}}{(1+\mu)^{1-\alpha}} = \Delta^{o}_{10}$
1	1,40	813,2	267	38,45	0,0267	0,0243	0,926	1,366	1,025	28,16	793
2	1,40	3210	813	95,10	0,0173	0,0243	1,356	1,578	0,887	60,27	3618
3	1,40	1626,4	271,8	64,50	0,0190	0,0243	1,264	1,530	0,915	42,16	1777
4	1,38	544	84,10	30,5	0,0270	0,0231	0,887	1,331	1,037	22,90	524,6
5	2,55	544	80,2	68,2	0,0220	0,0941	1,0976	2,794	0,913	24,41	595,8
6	2,61	1728	171,2	133,3	0,0190	0,0977	1,143	2,994	0,872	44,52	1982
7	11,30	685	51,4	231,4	0,0160	0,6255	0,9326	9,596	1,178	24,11	581,5
8	2,93	544	29,5	77,0	0,0215	0,1172	1,073	3,169	0,9245	24,30	588,4
9	2,65	544	12,20	67,37	0,0240	0,1001	1,059	2,807	0,944	24,0	576,3
10	2,65	544	16,05	68,10	0,0235	0,1001	1,0667	2,828	0,937	24,09	580,3
11	2,65	3210	43,6	224,1	0,0128	0,1001	1,2746	3,463	0,7652	64.71	4195
12	2,65	544	4,50	70,35	0,0220	0,1001	1,089	2,890	0,9169	24,34	593
13	2,67	1891	14,8	157,3	0,0155	0,1014	1,206	3,269	0,8168	48,12	2315
14	2,65	3321	19,6	206,5	0,0156	0,1001	1,207	3,242	0,8173	63,70	4063
15	8,85	544	2,76	169,7	0,0180	0,4764	0,940	7,764	1,140	21,86	477,3
16	2,65	3210	9,41	223,2	0,0129	0,1001	1,2719	3,454	0,7672	64,72	4184

$\nu = 1\text{x}10^{-6}\left(\frac{m^2}{s}\right)$ for water

TABLE 3: CALCULATED SUSPENSION VELOCITY $\Omega_{\mu\beta x}$ AND SUSPENSION HYDRAULIC GRADIENTS $I_{\mu\beta x}$ FOR FINITE VOLUMETRIC CONCENTRATIONS COMPARED WITH EXPERIMENTAL HYDRAULIC GRADIENTS $I_{\mu x\ experm}$.

1	2	3	4	5	6	7	8	9	10	11	12	13	
REF	1 + μ	$\Delta^{\circ}_{\mu o}$	$\delta_{\mu o}$	$\Omega_{\mu o}$	$I_{\mu o}$	$\left(\frac{\delta_{\mu o}}{\Delta^{\circ}_{\mu o}}\right)^2 = x$ or x	$\frac{\delta_{\mu o}}{\Delta^{\circ}_{\mu o}} = x^{\frac{1}{2}}$ or $x^{\frac{1}{2}}$	$\beta x^{\frac{1}{2}} = 1{,}11x^{\frac{1}{2}}$	$\frac{\Omega_{\mu o}}{(1-\beta x^{\frac{1}{2}})^{\frac{3}{2}}} = \Omega_{\mu\beta x}$	$\frac{I_{\mu o}}{(1-\beta x^{\frac{1}{2}})^{3}} = I_{\mu\beta x}$	$I_{\mu x}$ experim.	$\frac{I_{\mu\beta x}}{I_{\mu x\ experim}}$	NOTES
1	1,40	813,2	267	38,45 (2,70 ft/s)	0,0243	$\left(\frac{267}{813{,}2}\right)^2 = 0{,}1078$	0,328	1,11 x 0,328 = 0,364	75,83 (5,325 ft/s)	0,0945	0,0915	1,033	
2	1,40	3210	813	95,10 (6,67 ft/s)	0,0243	0,050	0,2236	0,248	145,8 (10,23 ft/s)	0,0571	0,0615	0,926	
3	1,40	1626	271,8	64,50 (4,53 ft/s)	0,0243	0,050	0,2236	0,248	98,92 (6,948 ft/s)	0,0571	0,0585	0,976	
						0,100	0,316	0,351	123,3 (8,66 ft/s)	0,08891	0,0875	1,016	
4	1,38	544	84,10	30,50 (0,653 m/s)	0,0231	0,050	0,2236	0,248	46,78 (1,00 m/s)	0,0541	0,057	0,949	
						0,100	0,316	0,351	58,32 (1,246 m/s)	0,08416	0,0860	0,979	
						0,150	0,387	0,4296	70.80 (1,513 m/s)	0,1239	0,1265	0,979	
						0,200	0,447	0,496	85,24 (1,822 m/s)	0,1797	0,167	1.076	
5	2,55	544	80,2	68,2 (4,78 ft/s)	0,0941	0,050	0,2236	0,248	104,60 (7,33 ft/s)	0,2214	0,235	0,942	
						0,100	0,316	0,351	130,4 (9,14 ft/s)	0,344	0,365	0,942	
						0,150	0,387	0,4296	158,75 (11,12 ft/s)	0,5100	0,540	0,944	
						0,200	0,447	0,496	196.6 (13,36 ft/s)	0,7351	–	–	Extrapolated

TABLE 3: *(continued)*

1 REF	2 1 + μ	3 $\Delta^\circ_{\mu o}$	4 $\delta_{\mu o}$	5 $\Omega_{\mu o}$	6 $I_{\mu o}$	7 $\left(\frac{\delta_{\mu o}}{\Delta^\circ_{\mu o}}\right)^2 = x$ or x	8 $\frac{\delta_{\mu o}}{\Delta^\circ_{\mu o}} = x^{\frac{1}{2}}$ or $x^{\frac{1}{2}}$	9 $\beta x^{\frac{1}{2}} = 1{,}11x^{\frac{1}{2}}$	10 $\frac{\Omega_{\mu o}}{(1-\beta x^{\frac{1}{2}})^{\frac{3}{2}}} = \Omega_{\mu\beta x}$	11 $\frac{I_{\mu o}}{(1-\beta x^{\frac{1}{2}})^{3}} = I_{\mu\beta x}$	12 $I_{\mu x}$ experim.	13 $\frac{I_{\mu\beta x}}{I_{\mu x}\text{ experim}}$	NOTES
8	2,93	544	25,9	77,0 (5,40 ft/s)	0,1172	0,018	0,134	0,149	98,09 (6,88 ft/s)	0,1902	0,200	0,951	
						0,037	0,192	0,2135	110,4 (7,74 ft/s)	0,241	0,259	0,931	
						0,079	0,281	0,312	134,9 (9,46 ft/s)	0,3598	0,370	0,972	
						0,155	0,394	0,437	182,5 (12,80 ft/s)	0,6570	0,600	1,095	
9	2,65	544	12,2	67,37 (4,73 ft/s)	0,1001	0,0402	0,200	0,2225	98,26 (6,900 ft/s)	0,2130	0,225	0,946	
						0,0862	0,2936	0,3259	121,72 (8,540 ft/s)	0,3268	0,340	0,961	
						0,1392	0,3733	0,4144	150,3 (10,54 ft/s)	0,498	0,510	0,976	
10	2,65	544	16,05	68,10 (4,778 ft/s)	0,1001	0,050	0,2236	0,248	104,44 (7,328 ft/s)	0,2355	0,2350	1002	
						0,100	0,3160	0,351	130.21 (9,135 ft/s)	0,3663	0,3750	0,977	
						0,150	0,3870	0,4296	158,08 (11,090 ft/s)	0,5393	0,550	0,972	
12	2,65	544	4,50	70,35 (4,936 ft/s)	0,1001	0,050	0,2236	0,248	107,9 (7,571 ft/s)	0,2355	0,224	1,051	
15	8,85	544	2,78	169,7 (11,91 ft/s)	0,4764	0,050	0,2236	0,248	260,27 (18,27 ft/s)	1,121	–	–	Extrapolated

FIG.1. GRAPHICAL REPRESENTATION OF HYDRAULIC CONDITIONS INSURING PARTICLE SUSPENSION

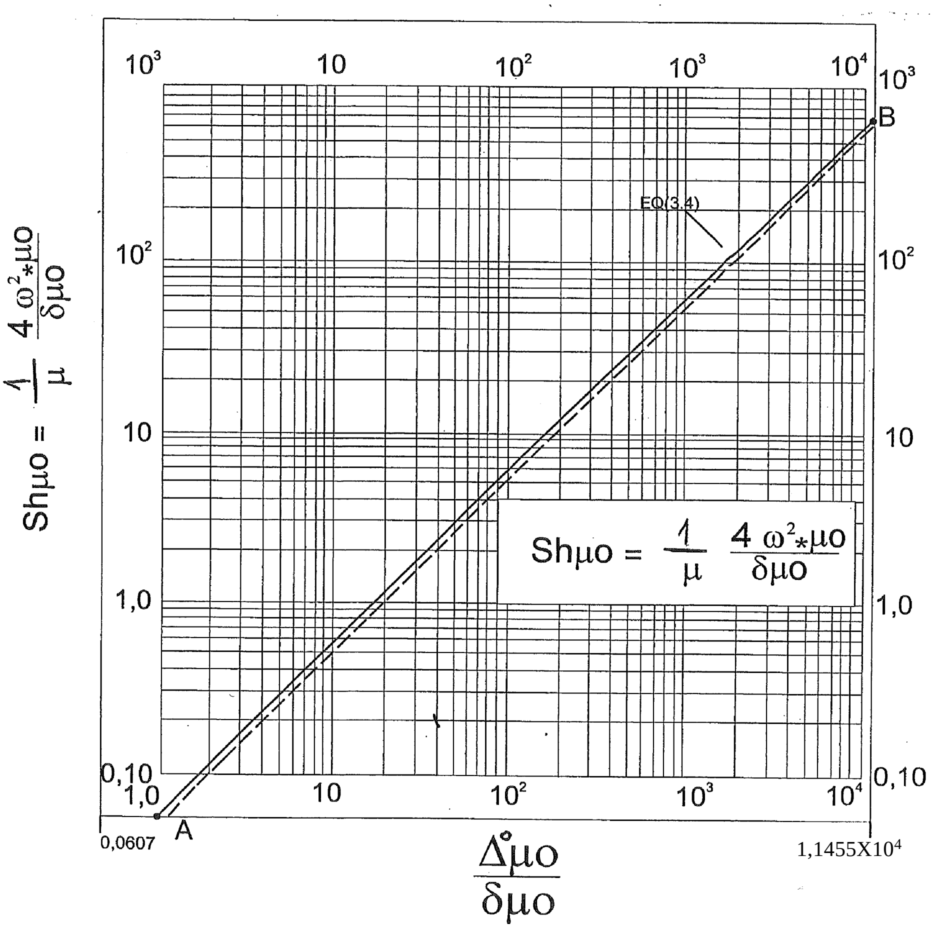


FIG.2. REPRESENTATION OF STREAM VELOCITIES AND HYDRAULIC GRADIENTS VERSUS THE VOLUMETIC CONCENTRATION X

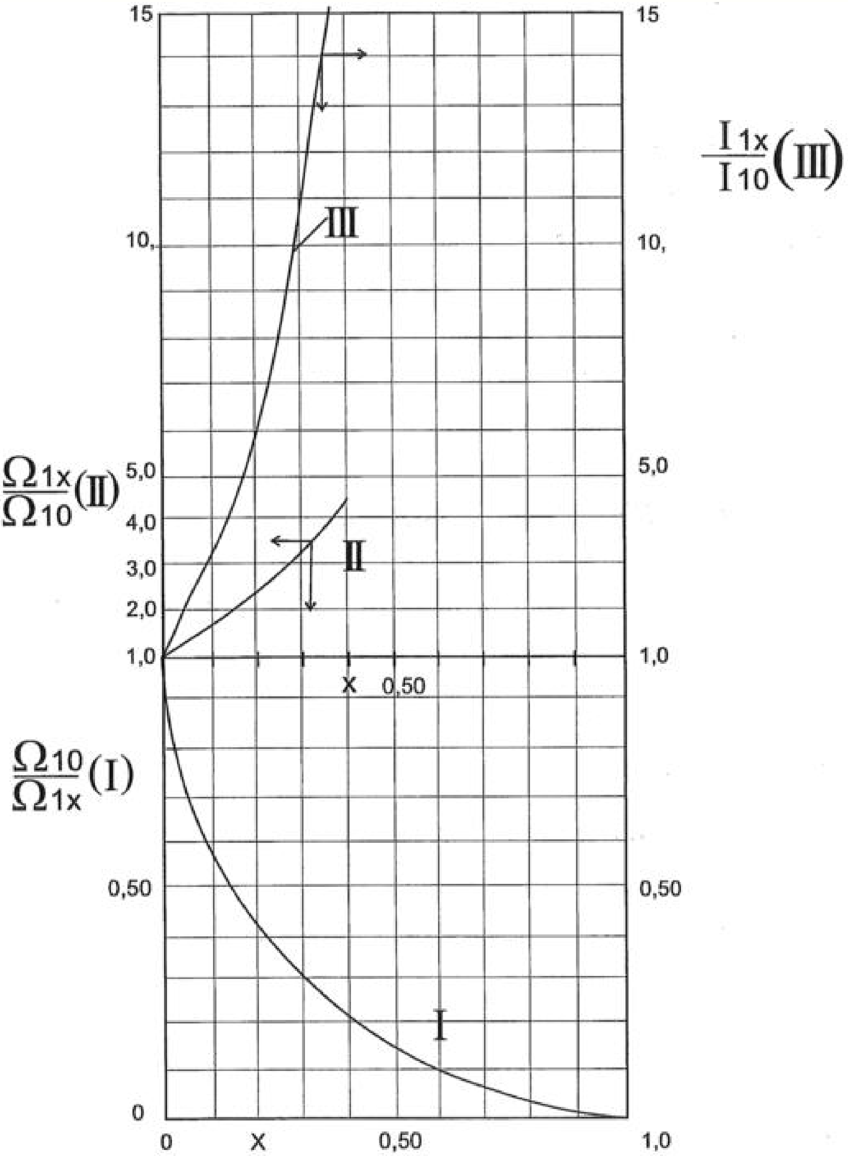


FIG.3. GRAPHICAL REPRESENTATION OF AUGMENTED PARTICLE DIAMETER EXPRESSING THE SAME VOLUMETRIC CONCENTRATION X OF A CYLINDRICAL PARTICLE

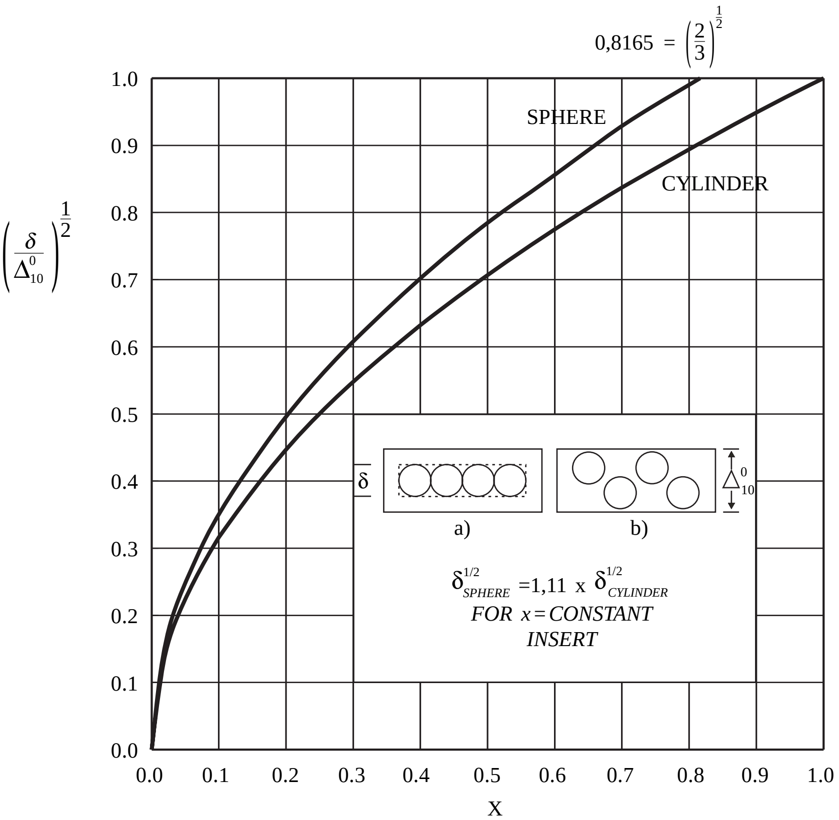
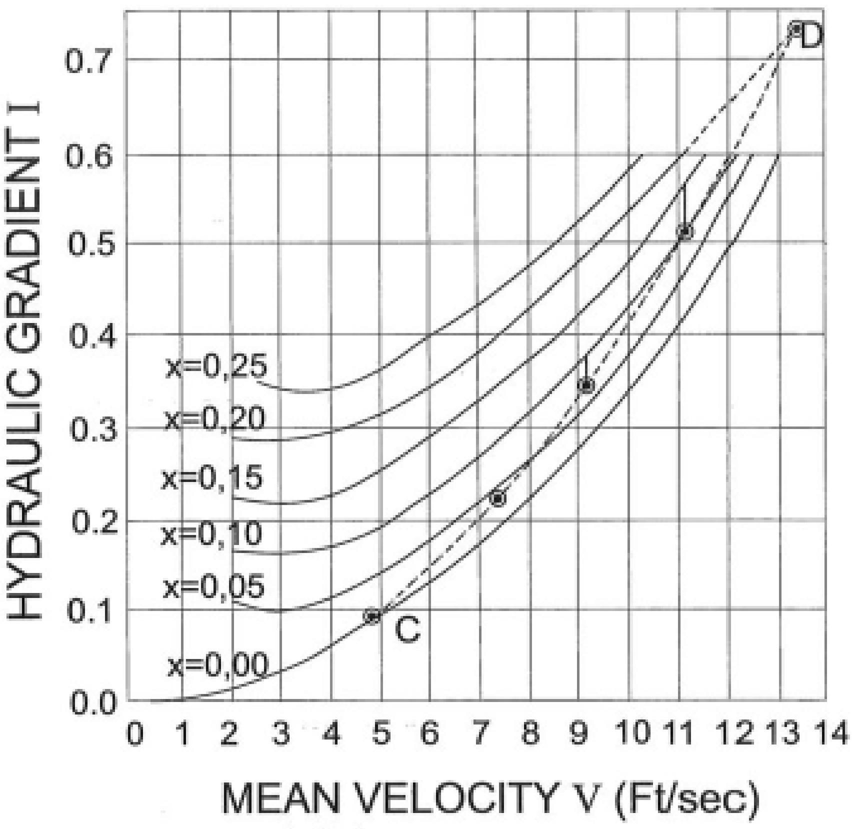


FIG.4. EXPERIMENTS OF HYDRAULICS TRANSPORT AS IN TABLE 1 REF. 5, REDRAWN BY AUTHOR WITH CALCULATED POINTS AS IN TABLE 3.



LIST OF FIGURES AND TITLES

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